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Legendre polynomials and its derivation

Hanan abushahma
h.abushahma@edu.misuratau.edu.ly
Mathematics Department- Faculty of Education- Misurata University

Abstract:
This paper presents Legendre polynomials which closely associated with physical phenomena and spherical geometry. Legendre polynomials are the simplest example of polynomial sets. Each polynomial set satisfies several recurrence formulas, and involved numerous integral relationships, also it forms the basis for series expansions resembling Fourier trigonometric series. Derivation of the Legendre differential equation is studied.
Keywords: Legendre polynomials, Binomial coefficients, Power Series.

كثيرات حدود لجندر واشتقاقها

حنان صالح أبوشحمة
قسم الرياضيات- كلية التربية- جامعة مصراتة

الملخص:
تهدف هذه الورقة البحثية إلى دراسة كثيرات حدود لجندر وعلاقتها بالظواهر الفيزيائية وقوي الجاذبية، فمتعددة حدود لجندر هي أبسط مثال على مجموعات متعددات الحدود التي تحقق العديد من صيغ التكرار وعلاقات التكامل وتعتبر أساس توسعات متسلسلة فورية المثلثية وكذلك تم دراسة اشتقاق معادلات لجندر التفاضلية.
الكلمات المفتاحية: كثيرات حدود لجندر- معاملات ذات الحدين- سلسلة القوى.

1. Introduction

In applied science almost all researchers encounter some special classical orthogonal functions such as Legendre, Hermite and Laguerre [Bell, 1967, Arfken, 1970] polynomials. Among these, Legendre polynomials have an extensive usage area, particularly in physics and engineering. For example, Legendre and Associate Legendre polynomials are widely used in the determination of wave functions of electrons in the orbits of an atom [Robert,

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1960, Hollas, 1992] and in the determination of potential functions in the spherically symmetric geometry [Jackson, 1962], etc. Also in nuclear reactor physics, Legendre polynomials have an extraordinary importance.
Legendre polynomials are closely associated with physical phenomena and spherical geometry . In particular, these polynomials first arose in the problem of expressing the Newtonian potential of a conservative force field in an infinite series involving the distance variables of two points and their included central angle. Other similar problems dealing with either gravitational potentials or electrostatic potentials also lead to Legendre polynomials.
There exists a whole class of polynomial sets which have many properties in common, Legendre polynomials represent the simplest example. Each polynomial set satisfies several recurrence formulas, which involved in numerous integral relationships, and basis of series expansions, which resembling Fourier trigonometric series when the sines and cosines are replaced by members of the polynomial set.
Factorials and Binomial coefficients:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + \frac{n(n-1)\dots(n-k+1)}{k!}a^{n-k}b^k + \cdots + b^n \qquad (1)$$

The coefficient of the general term in eq.(1) can be expressed more simply in terms of factorials by writing

$$\frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{k!\,(n-k)!}$$

For which we also introduce the notation

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}, \qquad n = 0,1,2,\dots, \qquad k = 0,1,\dots,n$$

We can write Eq.(1) more compactly as

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Example 1 : show that

$$\binom{-r}{k} = (-1)^k \binom{r+k-1}{k}$$

Solution:
we have

$$\binom{r}{k} = \frac{r(r-1)\dots(r-k+1)}{k!}$$

$$\therefore \binom{-r}{k} = \frac{-r(r-1)\dots(-r-k+1)}{k!}$$

$$= (-1)^k \frac{r(r+1)\dots(r+k-1)}{k!}$$

$$= (-1)^k \frac{r(r+k-1)(r+k-2)\dots(r+1)r}{k!}$$

$$= (-1)^k \binom{r+k-1}{k}$$

Definition1

Euler's integrative definition of a gamma function:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt; \quad x > 0 \qquad (4)$$

Legendre duplication formula

$$2^{2x-1}\Gamma(x)\Gamma\left(x+\tfrac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2x) \qquad (5)$$

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{(2n)!}{2^{2n} n!}\sqrt{\pi} \quad , \qquad n = 0,1,2,\dots \qquad (6)$$

Definition 2 (Power Series)

a power series, is an expression of the form

$$c_0 + c_1(x-a) + \dots + c_n(x-a)^n + \dots = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad (7)$$

Where the c's are constants and a is some fixed value.

Theorem (1) (Uniqueness)

If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ and $g(x) = \sum_{n=0}^{\infty} b_n(x-a)^n$
Both have nonzero radii of convergence, and $f(x) = g(x)$ wherever the two series converge, then $c_n = b_n,\ \ n = 0,1,2,\dots$.

Taylor Theorem (2)

Suppose f(x) is a defined function and continuous on the interval, [a,b] and suppose that the derivatives f^((n+1)) (x) where n a positive integer exists for |x-a|<ρ. So the function f(x) can be approximated to:

$$f(x) = \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} f^{(k)}(a) \qquad (8)$$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

$$f(x) = P_n(x) + R(x)$$

$$P_n(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a)$$

$$R(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \;, \quad R \to 0 \,, a < \xi < x$$

R(x) is called the remainnder.
Equation (8) is called Taylor Series for the function f. The special case of Eq. (8) that occurs when $a = 0$ is known as Maclaurin Series, i.e.,

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0) \qquad (9)$$

Example 2: Expand $f(x) = (1+x)^a$ using Maclaurin series, where a is a parameter not restricted to integer values.
Solution: Repeated differentiation of the function reveals that

$$f'(x) = a(1+x)^{a-1}$$
$$f''(x) = a(a-1)(1+x)^{a-2}$$
$$\vdots$$
$$f^{(n)}(x) = a(a-1)\cdots(a-n+1)(1+x)^{a-n}$$

Hence, by setting $x = 0$ in f and all its derivatives, we get series (2) so

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \cdots + \frac{a(a-1)\ldots(a-n+1)}{n!}x^n + \cdots$$

Which we can express more compactly in the form

$$(1+x)^a=\sum_{=0}^{\infty}\binom{a}{n}x^n \qquad (10)$$

2. The Generating Function

Among other areas of application, the subject of potential theory is concerned with the forces of attraction due to the presence of a gravitational field. Central to the discussion of problems of gravitational attraction is Newton's law of universal gravitation:

"Every particle of matter in the universe attracts each other particle with a force, whose direction is the line joining the two. And its magnitude products of the their masses and inversely as the square of their distance from each other."

The force field generated by a single particle is usually considered to be conservative. So that, there exists a potential function **V** such that the gravitational force F at a point of free space (i.e., free of point masses) is related to the potential function according to

$$\boldsymbol{F}=-\nabla V \qquad (11)$$

Where the minus sign is conventional. If r denotes the distance between a point mass and a point of free space, the potential function can be writing also,

$$V(r)=\frac{k}{r} \qquad (12)$$

k is a constant whose numerical value does not concern us. Because of spherical symmetry of the gravitational field, the potential function **V** depends only upon the radial distance **r**.

Valuable information on the properties of potential like Eq.(12) may be inferred from developments of the potential function into power series of certain types. In 1785, A.M. Legendre published his " Sur I'attraction des spheroids, in which he developed the gravitational potential Eq.(12) in a power series involving the ratio of two distance variables. He found that the coefficients appearing in this expansion were polynomials that exhibited interesting properties.

In order to obtain Legendre's results, let us suppose that a particle of mass $\boldsymbol{m}$ is located at point $\boldsymbol{P}$, which is a units from the origin of our coordinate system (see Fig 1). Let the point $\boldsymbol{Q}$ represent a point of free space $\boldsymbol{r}$ units from $\boldsymbol{P}$ and $\boldsymbol{b}$ units from the origin $\boldsymbol{O}$. For the sake of definiteness, let us assume $\boldsymbol{b}>\boldsymbol{a}$. Then, from the law of cosines, we find the relation

$$r^2 = a^2 + b^2 - 2ab\cos\emptyset \qquad (13)$$

Where $\emptyset$ is the central angle between the rays $\overline{\boldsymbol{0P}}$ and $\overline{\boldsymbol{OQ}}$. By rearranging the terms and factoring out $\boldsymbol{b^2}$, it follows that

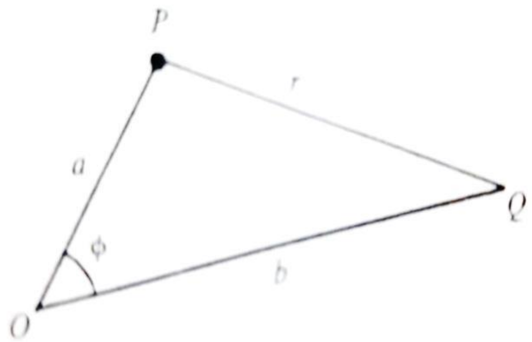

Figure 1

$$r^2 = b^2\left[1 - 2\frac{a}{b}\cos\emptyset + \left(\frac{a}{b}\right)^2\right], \quad a < b \qquad (14)$$

For more simplify of notation Eq.(14),

$$t = \frac{a}{b}, \qquad x = \cos\emptyset \qquad (15)$$

And thus, upon taking the square root,

$$r = b(1 - 2xt + t^2)^{1/2} \qquad (16)$$

Finally, substitution of Eq. (16) into Eq. (12) leads to the expression

$$V = \frac{k}{b}(1 - 2xt + t^2)^{-1/2}, \quad 0 < t < 1 \qquad (17)$$

For the potential function (for reasons that will soon be clear), refer to the function $w(x,t) = (1 - 2xt + t^2)^{-1/2}$ as the generating function of Legendre polynomials. Our task at this point is to develop $w(x,t)$ in a power series in the variable t.

3. Legendre polynomials

The binomial series

$$(1 - u)^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n u^n, \quad |u| < 1 \qquad (18)$$

Hence, by setting $u = t(2x - t)$, we find that

$$w(x,t) = (1 - 2xt + t^2)^{-1/2}$$
$$= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n t^n (2x - t)^n \quad \to (19)$$

Which is valid for $|2xt - t^2| < 1$. For $|t| < 1$, it follows that $|x| \leq 1$. The factor $(2x - t)^n$ is simply a finite binomial series, and thus Eq.(19) can further be expressed as

$$w(x,t) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n t^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k (2x)^{n-k} t^k$$

Or

$$w(x,t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{-\frac{1}{2}}{n} \binom{n}{k} (-1)^{n+k} (2x)^{n-k} t^{n+k} \qquad (20)$$

Since our goal is to obtain a power series involving powers of t to a single index, the change of indice $n \to n - k$ is suggested. Thus

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{n-k,k} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} A_{n-2k\,,k}$$

Eq. (20) can be written in the equivalent form;

$$w(x,t) = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{[n/2]} \left(\binom{-\frac{1}{2}}{n-k} \binom{n-k}{k} \right) (-1)^n \; (2x)^{n-2k} \right\} t^n \qquad (21)$$

The innermost summation in Eq.(21) is of finite length and therefore represents a polynomial in x, which happens to be of degree n. If we denote this polynomial by the symbol

$$P_n(x) = \sum_{k=0}^{[n/2]} \binom{-\frac{1}{2}}{n-k} \binom{n-k}{k} (-1)^n (2x)^{n-2k} \qquad (22)$$

then Eq.(21) leads to the intended result

$$w(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n \;, \quad |x| \leq 1, \quad |t| < 1 \quad (23)$$

Where $w(x,t) = (1 - 2xt + t^2)^{-1/2}$

The polynomials $P_n(x)$ are called the **legendre polynomials** in honor of their discoverer.

By recognizing that Eq.(3) and Eq.(6)

$$\binom{-\frac{1}{2}}{n} = (-1)^n \binom{n-\frac{1}{2}}{n}$$
$$= (-1)^n \frac{\Gamma\left(n+\frac{1}{2}\right)}{n!\,\Gamma\left(\frac{1}{2}\right)}$$
$$= \frac{(-1)^n (2n)!}{2^{2n}(n!)^2}$$

It follows that the product of binomial coefficients in Eq.(22) is

$$\binom{-\frac{1}{2}}{n-k}\binom{n-k}{k} = \frac{(-1)^{n-k}(2n-2k)!}{2^{2n-2k}(n-k)!\,k!\,(n-2k)!} \qquad (25)$$

And hence, Eq.(22) becomes

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!\, x^{n-2k}}{2^n k!\,(n-k)!\,(n-2k)!} \qquad (26)$$

The first few Legendre polynomials are listed in Table (1)

Making an observation, we note that when n is an even number the polynomial $P_n(x)$ 1is an even function, and when n is odd the polynomial is an odd function too.

Therefore,

$$P_n(-x) = (-1)^n P_n(x), \quad n = 0,1,2 \ldots. \qquad (27)$$

The graphs of $P_n(x), n = 0,1,2,3,4$ are sketched in Fig. 2 over the interval $-1 \leq x \leq 1$.

Table (1) Legendre polynomials

$$P(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$
$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

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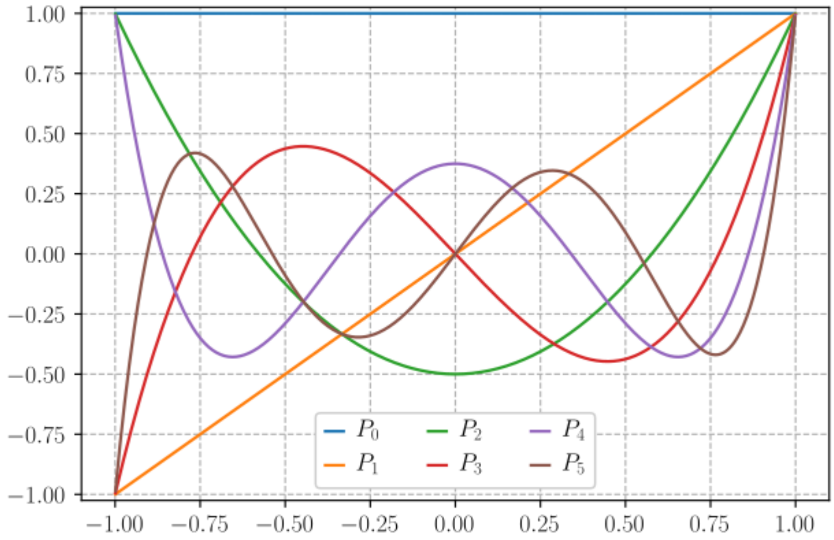

Figure 2

Returning now to Eq.(27) with x =cos ∅ And $t = a/b$, we find that, the potential function has the series expansion

$$V = \frac{k}{b}\sum_{n=0}^{\infty} P_n\,(cos\,\emptyset)\;\;(\frac{a}{b})^n \qquad a < b \qquad (28)$$

In terms of the argument $cos\ \emptyset$,the Legendre polynomials can be expressed

as trigonometric polynomials of the form shown in Table 2 .
In Fig.3 the first few polynomials $P_n\ (cos\,\emptyset)$ are plotted as a function

4. Special Values and Recurrence Formulas.

The Legendre polynomials are rich in recurrence relations and identities. Central to the development of many of these is the generating- function

Table (2) Legendre trigonometric polynomials

$$P_0\ (cos\,\emptyset) = 1$$

$$P_1(\cos\emptyset) = \cos\emptyset$$
$$P_2(\cos\emptyset) = \frac{1}{2}(3\cos^2\emptyset - 1)$$
$$= \frac{1}{4}(3\cos 2\emptyset + 1)$$

$$P_3(\cos\emptyset) = \frac{1}{2}(5\cos^3\emptyset - 3\cos\emptyset)$$
$$= \frac{1}{8}(5\cos 3\emptyset + 3\cos\emptyset)$$

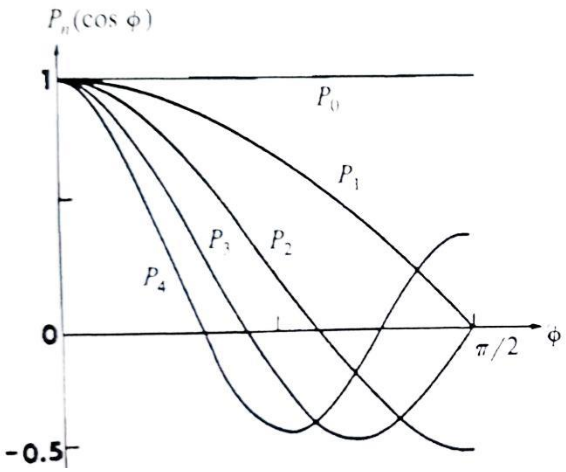

Figure 3

Relation

$$(1 - 2xt + t^2)^{\frac{-1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \qquad |x| \le 1,\ |t| < 1 \qquad (29)$$

Special values of the Legendre polynomials can be derived directly from Eq.(29) by substituting particular values for x .For example, the substitution of x =1 yields

$$(1 - 2ty + t^2)^{-\frac{1}{2}} = (1 - t)^{-1} = \sum_{n=0}^{\infty} P_n(1)t^n \qquad (30)$$

However, we recognize that $(1 - t)^{-1}$ is the sum of a geometric series, so that Eq.(30) is equivalent to

$$\sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} P_n(1)t^n \qquad (31)$$

Hence, from uniqueness theorem of power series (theorem1), we can compare like coefficients of t^n in Eq.(31) to deduce the result

$$P_n(1) = 1, \qquad n = 0,1,2,\dots \qquad (32)$$

Also from Eq.(27) we see that

$$P_n(-1) = (-1)^n, \quad n = 0,1,2,\dots \qquad (33)$$

The substitution of $x = 0$ into Eq.(29) leads to

$$(1+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(0)t^n \qquad (34)$$

But the term on the left-hand side has the binomial series expansion

$$(1+t^2)^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} t^{2n} \qquad (35)$$

Comparing terms of the series on the right in Eq.(34) and Eq.(35), we note that Eq.(35) has only even power of t . Thus we conclude that $P_n(0) = 0$, For $n = 1,3,5,\dots,$
or equivalently,

$$P_{2n+1}(0) = 0, \quad n = 0,1,2,\dots \qquad (36)$$

Since all odd terms in Eq.(34) are zero, we can replace n by $2n$ in the series and compare with Eq.(35), from which we deduce

$$P_{2n}(0) = \binom{-\frac{1}{2}}{n} = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}, \quad n = 0,1,2,\dots \qquad (37)$$

Where we are recalling (24)

Remark: Actually, Eq.(36) could have been deduced from the fact that $P_{2n+1}(x)$ is an odd (continuous) function, and therefore must necessarily pass though the origin.

In order to obtain the desired recurrence relation, we first make the observation that the function $w(x,t) = (1-2xt+t^2)^{-1/2}$ satisfies the derivative relation

$$(1-2xt+t^2)\frac{\partial w}{\partial t} + (t-x)w = 0 \qquad (38)$$

Direct substitution of the series Eq.(23) for $w(x,t)$ into Eq.(38) yields

$$(1-2xt+t^2)\sum_{n=0}^{\infty} nP_n(x)t^{n-1} + (t-x)\sum_{n=0}^{\infty} P_n(x)t^n = 0$$

Carrying out the indicated multiplications and simplifying gives us

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} - 2x\sum_{n=0}^{\infty} nP_n(x)t^n + \sum_{n=0}^{\infty} nP_n(x)t^{n+1} + \sum_{n=0}^{\infty} P_n(x)t^{n+1}$$
$$-x\sum_{n=0}^{\infty} P_n(x)t^n = 0 \qquad (39)$$

We now wish to change indices so that powers of t are the same in each summation. We accomplish this by leaving the first sum in Eq.(39) as it is, replacing n with $n-1$ in the second and last sums, and replacing n with $n-2$ in the remaining sums; thus Eq.(39) becomes

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} - 2x\sum_{n=1}^{\infty} (n-1)P_{n-1}(x)t^{n-1} + \sum_{n=2}^{\infty} (n-2)P_{n-2}(x)t^{n-1}$$
$$+\sum_{n=2}^{\infty} P_{n-2}(x)t^{n-1} + x\sum_{n=1}^{\infty} P_{n-1}(x)t^{n-1} = 0$$

Finally, combining all summations, we have

$$\sum_{n=2}^{\infty} [nP_n(x) - 2x(n-1)P_{n-1}(x) + (n-2)P_{n-2}(x) + P_{n-2}(x)$$
$$- xP_{n-1}(x)]t^{n-1} + P_1(x) - xP_0(x) = 0 \qquad (40)$$

But $P_1(x) - xP_0(x) = x - x = 0$, and the validity of Eq.(40) demands that the coefficient of t^{n-1} in Eq.(31) be zero for all x. Hence. After simplification we arrive at

$$nP_n(x) - (2n-1)xP_{n-1}(x) + (n-1)P_{n-2} = 0, \qquad n = 2,3,4,\ldots$$

Or, replacing n by $n+1$, we obtain he more conventional form

$$(n+1)P_{n+1}(x) - (2n-1)xP_n(x) + nP_{n-1} = 0 \qquad (41)$$

Where $n = 0,1,2,\ldots$

We refer to (41) as a $three - term\ recurrence\ formula$, since it forms a connecting relation between three successive Legendre polynomials. One of the primary uses of (41)in computations is to produce higher-order Legendre polynomials from lower-order ones by expressing them in the form

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$$P_{n+1}(x) = \left(\frac{2n+1}{n+1}\right) x p_n\,(x) - \left(\frac{n}{n+1}\right) p_{n-1}(x) \qquad (42)$$

Where $n = 1,2,3, \ldots$

In practice, (42) is generally preferred to (26) in making computer calculation when several polynomials are involved.

A relation similar to (41) involving derivatives of the Legendre polynomials can be derived in the same fashion by first making the observation that $w(x,t)$ satisfies

$$(1 - 2xt + t^2)\,\frac{\partial w}{\partial t} - tw = 0 \qquad (43)$$

Where this time the differentiation is, with respect to x. Substituting the series for $w(x,t)$ directly into (43) leads to

$$(1 - 2xt + t^2) \sum_{n=0}^{\infty} p'_n(x) t^n - \sum_{n=0}^{\infty} P_n(x) t^{n+1} = 0$$

Or, after carrying out the multiplications.

$$\sum_{n=0}^{\infty} P'_n(x) t^n - 2x \sum_{n=0}^{\infty} P'_n(x) t^{n+1} + \sum_{n=0}^{\infty} P'_n(x) t^{n+2} - \sum_{n=0}^{\infty} P_n(x) t^{n+1} = 0 \quad (44)$$

Next, making an appropriate change of index in each summation, we get

$$\sum_{n=2}^{\infty} [P'_n\,(x) - 2x\,P'_{n-1}\,(x) + P'_{n-2}\,(x) - P_{n-1}(x)\,] t^n = 0 \quad (45)$$

Where all terms outside this summation add to zero. Thus, by equating the coefficient of t^n to zero in (45),we find

$$P'_n\,(x) - 2xP'_{n-1}\,(x) + P'_{n-2}\,(x) = 0, \quad n=2,3,4\ldots\ldots$$

Or ,by a change of index,

$$P'_{n+1}\,(x) - 2xP'_n(x) + P'_{n-1}(x) - P_n(x) = 0 \qquad (46)$$

for n = 1,2,3,………..

Certain combinations of Eq.(31) and Eq.(46) can lead to further recurrence relations. For example, suppose we first differentiate (41),i.e.,

$$(n+1)P'_{n+1}(x) - (2n+1)P_n(x) - (2n+1)xP'_n(x) + nP'_{n-1}(x) = 0 \qquad (47)$$

From Eq.(46) we find

$$P'_{n-1}(x) = P_n(x) + 2xP'_n(x) - P'_{n+1}(x) \quad (48a)$$

$$P'_{n+1}(x) = P_n(x) + 2xP'_n(x) - P'_{n-1}(x) \quad (48b)$$

And the successive replacement of $P'_{n-1}(x)$ and $P'_{n+1}(x)$ in (47) by (48a) and (48b) leads to the two relations

$$P'_{n+1}(x) - xP'_n(x) = (n+1)P_n(x) \quad \to (49a)$$
$$xP'_n(x) - P'_{n-1}(x) = nP_n(x) \quad \to (49b)$$

The addition of Eq.(39a) and Eq.(39b) yields the more symmetric formula

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x) \qquad (50)$$

Finally replacing n by $n-1$ in Eq.(49a) and them eliminating the term $P'_{n-1}(x)$ by use of Eq.(49b), we obtain

$$(1-x^2)P'_n(x) = nP_{n-1}(x) - nxP_n(x) \qquad (51)$$

This last relation allows us to express the derivative a Legendre polynomial in terms of Legendre polynomials.

5. Legendre's Differential Equation

All the recurrence relation that we have derived thus far involve successive Legendre polynomials. We may well wonder if any relation exists between derivatives of the Legendre polynomials and Legendre polynomials of the same index. The answer is in the affirmative, but to derive this relation we must consider second derivatives of the polynomials.

By taking the derivative of both sides of the Eq.(51), we get

$$\frac{d}{dx}[(1-x^2)P'_n(x)] = nP'_{n-1}(x) - nP_n(x) - nxP'_n(x)$$

And then, using Eq.(49b) to eliminate $P'_n(x)$, we arrive at the derivative Relation

$$\frac{d}{dx}[(1-x^2)P'_n(x)] + n(n+1)P_n(x) = 0 \qquad (52)$$

Which holds for $n = 0,1,2,\ldots$ Expanding the product term in Eq.(52) yields

$$(1-x^2)\, P''_n(x) - 2xP'_n(x) + n(n+1)\, P_n(x) = 0 \qquad (53)$$

And thus we deduce that the Legendre polynomials $y = P_n(x)$ $, n = 0,1,2,\ldots$ is a solution of the linear second-order DE

$$(1-x^2)\, y'' - 2xy' + n(n+1)\, y = 0 \qquad (54)$$

Called *Legendre 's differential equation*.

Perhaps the most natural way in which Legendre polynomials arise in practice is as solutions of Legendre's equation. In such problems the basic

Model is generally a partial differential equation. Solving. The partial DE by the separation –of– variables technique leads to a system of ordinary Des And sometimes one of these is Legendre’s DE. This is precisely the case, for example, in solving for the steady- state temperature distribution (independent of the azimuthal angle) in a solid sphere.

Remark: any function $f_n(x)$ that satisfies Legendre's equation, i.e.

$$(1 - x^2)f_n''(x) - 2xf_n'(x) + n(n+1)f_n(x) = 0$$

Will also satisfy all previous recurrence formulas given above, provided that $f_n(x)$ is properly normalized.

6. **Conclusion**

Some special functions are the basis for many mathematics sciences, especially those related to gender functions that have several properties. Therefore, they are studied in many branches of mathematics such as ordinary and partial differential equations and branches of physics.

This research is considered as a summary of gender differential equations and the basis of their formation and legendre polynomials and derivation of differential legendre equation.

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